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BACTERICIDAL PROPERTIES OF THE PRODUCTS OF RADIUM EMANATION

By John Albert Marshall

College of Dentistry, University of California

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The following abstract deals with a description of the use of radium therapy in periapical infections,[1] and supplements a preliminary note[2] in which it was pointed out that solutions of the products of radium emanation possess distinct bactericidal properties. Additional data have been compiled but the analysis of them clearly indicates that with the present-day knowledge it is impossible to state definitely and conclusively why the products of the emanation produce such characteristic effects.

Although it was reported in 1909[3] that radium exerts an inhibitory influence on pathogenic and non-pathogenic organisms, no further work was done in this field until ten years later, when the former observations were confirmed.[4] Bacteria of various types were exposed to radium emanation with the result that a distinct inhibitory effect upon the cultures was observed.

In this present experiment a *solution* of the products of radium emanation is employed. The emanation tubes are usually discarded by the laboratories after they have undergone disintegration in respect to their content of gamma ray. However, they still contain in varying proportions radium A, B, C, and D. The effects which have been obtained are presumably not due to the hard gamma ray, but result in all probability from the action of the softer beta ray.

The radioactive solution is prepared as follows: a "de-emanated" emanation tube is placed in a sterilized mortar of about 20-cc. capacity, covered with Ringer's solution and then crushed with a pestle. On account of the rapidity with which the radio-activity of this solution diminishes, it is advisable to crush the tubes immediately before use. This solution is carried directly to the infected area by means of sterile dressings. In the case of chronic or acute apical abscesses, found about the roots of

teeth, the radioactive liquid is forced into the infected area through the pulp canal of the tooth and allowed to remain for 24 hours. Subsequently the dressings are removed, the canals tested for sterility and finally filled. With two possible exceptions there has been no evidence, after three years time, of succeeding soreness or pain. The individual dosage in terms of millicuries of gamma radiation has varied from .046 to .138.

Failures in this method of treatment have been recorded but the fact remains that sterile tests have been obtained more often by using the radioactive solution than in similar experiments with other antiseptic agents.

Summaries of experimental and clinical data are presented which indicate the variations in the pathology, age of patient, etiology, treatment, clinical result and laboratory findings.

In many instances tubes have been used in which de-emanation had proceeded to such a degree that it was not possible to obtain a reading by the electroscope. These have given similar results insofar as bactericidal action is concerned, but they have not caused the soreness which sometimes accompanies the use of solutions containing a recorded amount of gamma rays.

In general the cases selected have been those in which previous treatments were without satisfactory result. Dichloramin T, formocresol beechwood cresote, phenol, and other antiseptics of a similar nature had been used without producing a noticeable change in condition. In fact, the greater majority of the cases carried a history of unsuccessful treatment. Most of them yielded to the action of the radioactive solution.

The precautionary measures which are to be observed are naturally those referring to dosage and technique of application. Severe radium burns resulted in monkeys when emanation tubes containing eighteen millicuries were sealed into pulp canals of central incisors. These burns were so extreme that the hard palate and orbit were involved. It is believed to be impracticable to use the radium except as a solution of the solids. The tubes should be thoroughly crushed under Ringer's solution exactly at the moment of use, and it has been customary to have three or four patients prepared and ready at one time in order to secure the greatest economy of material. The limited availability of de-emanated tubes except at emanation laboratories is a serious drawback to an extensive application of this therapy. However, it is planned, if funds become available, to further extend these researches to include not only additional data upon dosage but also to report upon the application of other radioactive preparations which may be found to be more plentifully distributed.

[1] Marshall, John A., "Radium in Periapical Infections" (*in press*).

[2] Marshall, John A., "Bactericidal Action of Solutions of Radium Emanation," *Calif. State Med. J.*, Jan. 1922.

[3] Iridell and Minett, "Notes of the Effect of Radium in Relation to Some Pathogenic and Non-Pathogenic Organisms," *Lancet*, 1909, **1**, p. 1445.

[4] Lequeux and Chrome, "Action of Radium on Bacteria," *Arch. Memo. Obst. gynec.*, **3**, p. 698, Dec. 1919.

THE JADES OF MIDDLE AMERICA[1]

BY HENRY S. WASHINGTON

GEOPHYSICAL LABORATORY, CARNEGIE INSTITUTION OF WASHINGTON

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The following pages present some of the results of a study of jade beads, disks, and other small objects, found in a *cenote*, or sacred natural well, at the ancient Maya city of Chichen Itza, in northern Yucatan, the petrographic study of which was kindly entrusted to me by Prof. A. M. Tozzer of Harvard University. The final results will be incorporated in a forthcoming archaeological monograph on the contents of the cenote, under the editorship of Prof. Tozzer. In connection with this study, I have also been able to examine some ancient Mexican beads, through the kindness of Dr. L. Salazar, Director of the Geological Survey of Mexico, and some jade objects from Copan, Honduras, which were kindly placed at my disposal by Dr. Sylvanus G. Morley. I am deeply indebted to Dr. H. E. Merwin, of this Laboratory, for numerous optical determinations.

There is great variety in all the characters, chemical, mineral, and physical, of the Chichen Itza and other American jades. There are two dominant colors: green and gray. Some objects are wholly of a deep, rather yellowish green, like Ridgway's "pistachio green," or of a grayish green, like Ridgway's "pea green," the first being used for the choicest objects. The grays vary from almost pure white, through ash-, gull-, and dove-gray, to rather dark gray, most of the gray specimens being mottled in lighter and darker shades. Another and common type is mottled white (or very light gray) and a clear, bright, grass-, or emerald-green. A few obsidian beads were also found at Chichen Itza.[2]

The texture varies from decidedly coarse-grained, as are most of the pistachio-green and mottled green and white specimens, to very fine grained, as are most of the pea-green specimens; the gray specimens are mostly moderately fine-grained. The coarse-grained pieces show glistening cleavage surfaces of pyroxene on the fracture. Nearly all these objects have a high to very high polish, which is most marked on the coarser and the pistachio-green varieties. The coarse varieties are rather translucent, as are some of the fine-grained pea-green beads; the whites and grays are generally opaque.

A few beads are dead white, of dull luster, and slightly crumbly. These have evidently been subjected to the action of fire, and the effect of this is clearly seen in one or two long beads which have been heated to the point of softening, so that they are bent and warped to such an extent that the longitudinal perforation is no longer straight.

The coarse-grained, translucent, green or green and gray varieties have a hardness of about $6^3/_4$, while the mottled gray and white kinds are about $6^1/_2$. None of this American jade shows the extreme toughness which is so characteristic of most jade, especially nephrite.

Many density determinations have been made, representing the various color and textural varieties, from Chichen Itza, Copan, and Mexico. The density varies from 3.335, about that of pure jadeite, to 2.667, about that of oligoclase. Some characteristic densities are given later. The numerous determinations by Hallock, published by Clarke and Merrill,[3] also show the wide variation in density.

In thin section most of the coarser specimens, whether wholly green, mottled green and white, or wholly white or gray, are seen to be composed entirely of anhedral or subhedral grains of pyroxene, no other mineral being present. The pyroxene grains are somewhat cracked, but there is little other evidence of crushing. The megascopically white and gray material is perfectly clear and colorless, while the green specimens, and the green portions of those which are mottled, are of a light, slightly yellowish green, with slight pleochroism. The green color is obviously not caused by the aegirite molecule. In the green-white mottled specimens, the color is distributed like a stain, covering small irregular areas, and fading out gradually in all directions, so that some grains near the border are partly green and partly colorless. In some pieces the green color is seen megascopically to be deeper along fine cracks, indicating the presence of a chromium-bearing solution. The purely pyroxenic specimens, it may be said, are those of the higher densities.

In other specimens, of the lower densities, and chiefly among the mottled green and white, mottled white and gray, and the fine-grained light pea-green varieties, there is more or less feldspar, which appears as a water-clear albite, in irregular interstitial areas or a granular mass. This albite very rarely shows twinning. The relative amount of albite (or oligoclase) varies within wide limits, from a few shreds and patches to the greater part of the section. As the amount of albite increases the pyroxene grains become smaller, more prismatic, and more scattered, until in the most albitic specimens the small percentage of pyroxene forms small prisms scattered through a matrix of albite grains.

Apart from the pyroxene and albite, no other minerals are present,

except for a very few small shreds of muscovite and an occasional very minute magnetite grain.

The pyroxene is of very variable composition, but is composed essentially of jadeite and diopside, the proportions of which vary within wide limits, from jadeite : diopside = 21 : 1 in No. 6, to about equal amounts in the Tuxtla statuette,[3] and even 1 : $2^1/_2$ in the Copan pebble.[4] Jadeite can evidently take up considerable amounts of diopside in solid solution. The material of the Tuxtla statuette, with about equal amounts of each, may be regarded as an end member of the series, and for this the name *tuxtlite* (pronounced tushtlite) is proposed. It may be a double salt, analogous to diopside or dolomite. Its optical characters (as determined by Merwin) are given in the paper cited, as well as its density (determined by Adams) and its chemical composition. From the data given later in table I, it would appear that the diopside-jadeite can take up albite and anorthite in solid solution to a certain extent, although according to Bowen,[5] there is no experimental evidence of the solution of plagioclase in typical diopside, $CaMgSi_2O_6$.

Some of the chemical analyses made by me of various Middle American jades, with two made by other analysts, are presented in table I. The densities and the optical properties of the pyroxene were determined on the same fragment which was used for the analysis. For the optical determinations I am deeply indebted to Dr. H. E. Merwin. With each analysis is given the composition calculated in terms of the four molecules: jadeite, diopside (including in some cases very small amounts of hypersthene, or wollastonite and a little excess ferric oxide), albite (including minute amounts of orthoclase), and anorthite. After allotting alumina to potash and soda, the invariable small excess was allotted to lime in anorthite. Diopside was then calculated, the usual slight excess of magnesia and ferrous oxide over lime forming very small amounts of hypersthene. The residual silica was, in every case, more than enough to form jadeite and less than enough to form albite; so it was distributed between these two molecules by the usual equations used in calculating the norm.

The first five specimens are shown by the microscope to be quite monomineralic, composed entirely of pyroxene, and without any feldspar, although No. 5 shows a very few small shreds of muscovite. The section of No. 6 shows a little albite, in small interstitial grains; and from this on to No. 12 (inclusive) the amount of visible albite steadily increases. The jade of the Copan pebble differs markedly from the others, in texture and composition of the pyroxene; it has been described elsewhere.[4] No. 13 is the analysis of a small bead given me by Dr. Salazar, which is remarkable as being related to nephrite. It is composed of a densely felted mixture of pyroxene and zoisite, the latter as determined by Merwin.

TABLE I

ANALYSES OF MIDDLE AMERICAN JADES

	1	2	3	4	5	6	7	8	9	10	11	12	13
SiO_2	55.50	58.70	57.92	59.03	59.80	59.93	59.60	60.99	63.47	64.64	62.64	67.16	46.00
Al_2O_3	12.33	22.55	19.74	23.26	21.79	23.38	21.85	22.20	20.76	18.83	14.92	18.81	18.64
Fe_2O_3	1.41	0.48	1.08	0.41	0.20	0.44	0.48	....	1.27	0.46	0.60	0.27	0.92
FeO	1.33	0.53	0.48	0.42	0.66	0.41	0.58	0.65	.....	0.49	1.25	0.32	2.24
MgO	8.72	2.14	3.34	1.31	2.25	0.71	2.57	0.96	1.11	1.87	4.31	0.57	11.09
CaO	12.76	3.07	4.67	0.92	2.50	1.07	3.29	1.28	1.16	2.62	5.92	1.33	17.49
Na_2O	6.94	12.87	11.28	13.31	12.50	13.34	11.78	13.04	11.98	11.25	8.78	11.21	0.64
K_2O	0.25	0.15	0.25	0.31	0.27	0.29	0.20	0.21	0.34	0.23	0.23	0.22	0.08
H_2O+	0.10	0.12	0.80	0.65	0.32	0.42	.....	0.74	0.36	0.16	1.27	0.14	2.38
H_2O-	0.20	.....	....	0.09	.....	0.21	.....	....	.....	.....	....	.....	0.21
TiO_2	none	none	none	none	0.02	none	none	n.d.	n.d.	none	none	none	0.05
Cr_2O_3	none	0.08	0.17	0.26	0.05	0.09	.....	n.d.	n.d.	0.07	none	none	....
MnO	0.05	none	none	none	none	none	none	n.d.	trace	none	none	none	trace
	99.59	100.69	99.73	99.97	100.36	100.29	100.35	100.07	100.45	100.46	99.92	100.02	99.74
Sp. gr.	3.270	3.183	3.312	...	3.052	3.237	3.196	....	2.832	2.737	2.934	2.667	3.161
C°	....	17.9°	19.2°	...	23.9°	21.3°	22.8°	....	....	23°	19.4°	24.4°	24.4°
Jd	45.15	81.69	65.93	78.11	70.50	73.06	62.24	64.45	40.55	25.15	10.25	4.05	....
Di	51.51	11.54	18.72	5.27	11.58	3.48	12.01	5.60	6.64	11.10	25.78	4.80	....
Ab	1.67	3.45	12.54	13.82	15.44	20.10	19.65	28.27	51.14	63.47	63.40	90.87	....
An	1.67	3.32	2.81	2.80	2.48	3.36	6.10	1.68	1.67	0.28	0.57	0.28	....

1. Tuxtlite. Tuxtla statuette. Coarse-grained; pea-green. Washington analyst. *Proc. U. S. Nat. Mus.*, **60**, Art. 14, 1922 (5).
2. Chichen Itza. Spheroidal bead; coarse, dull white. Washington analyst.
3. Chichen Itza. Spherical bead; coarse, mottled green and white. Washington analyst.
4. Chichen Itza. Flat bead; coarse, mottled green and white. Washington analyst.
5. Chichen Itza. Long, cylindrical bead; mottled gray and green. Washington analyst.
6. Chichen Itza. Long, prismatic bead; coarse, light gray. Washington analyst.
7. Chichen Itza. Spheroidal bead; very coarse, mottled dark gray and green. Washington analyst.
8. "Jadeite," Mexico. Color ? G. W. Hawes analyst. Dana, *System of Mineralogy*, 1892, p. 370, No. 22.
9. "Jade," Mexico. Green and gray. P. T. Walden analyst. Penfield in Bishop, Jade Book, Vol. I, 1906. MgO given as 11.11 "Albite present."
10. Chichen Itza. Cylindrical bead; fine-grained, gray-green. Washington analyst.
11. Copan. Worked pebble; coarse, light pea-green. Washington analyst. *Jour. Wash. Acad.*, **12**, 1922 (p. 390).
12. Techalquilla, Mexico (from Guerrero?). Small triangular bead; fine-grained, tea-green. Washington analyst.
13. "Nephrite?" Atzacapotzalco, Mexico (from Guerrero?). Spheroidal bead; mottled sage- and tea-green. Washington analyst.

The general absence of titanium and manganese is to be noted, while chromium is present in easily determinable amounts in all the specimens in which it was looked for, except No. 1. Considerable chromium is present in some of the purely gray specimens, as No. 6), which shows no trace of green color.

The analyses are arranged in the order of decreasing pyroxene (diopside-jadeite) and increasing feldspar. The density varies quite regularly with this, but it is also influenced by the amount of jadeite relative to diopside in the pyroxene, those higher in jadeite showing the higher density. The relative amounts of jadeite and diopside vary widely and without any very marked regularity.

The refractive indices of some of the pyroxenes, determined by Dr. Merwin, are given in table II In this the relative amounts of jadeite and diopside (calculated to 100%) are given, the amount of albite (and anorthite) held in solid solution in the pyroxene being neglected. The serial progression in the values for α and γ as well as for the birefringence $\gamma - \alpha$,[6] as the pyroxene changes from nearly pure jadeite to pure diopside, is evident. There are slight irregularities at the extremes, and it would appear that the influence of the "occult" albite is simply to lower the refractive indices. This matter is to be studied further by Merwin and me.

TABLE II

REFRACTIVE INDICES OF THE DIOPSIDE-JADEITE SERIES

	Jd	Di	α	γ	$\gamma - \alpha$
Jadeite (Tibet)[7]	98.68	1.32	1.655	1.667	0.012
Chichen Itza (No. 6)	95.45	4.55	1.651	1.667	0.016
Chichen Itza (No. 4)	93.69	6.31	1.650	1.668	0.018
Chichen Itza (No. 2)	87.62	12.38	1.655	1.672	0.017
Chichen Itza (No. 3)	77.89	22.11	1.660	1.678	0.018
Tuxtla (No. 1)	46.60	53.40	1.666	1.688	0.022
Copan pebble (No. 11)	28.45	71.55	1.665	1.693	0.028
Diopside (artificial)[8]	none	100.00	1.664	1.694	0.030

The feldspars show much less variation, which is not quite regular, but in general the feldspar becomes more albitic with increase of the amount of feldspar in the specimen. From what has been said, it is evident that the pyroxene of the wholly pyroxenic specimens contains considerable "occult" feldspar, both albite and anorthite, held in solid solution. No. 5 shows no feldspar in the thin section, although about 18% of $Ab_{86}An_{14}$ must be calculated as present. No. 6, however, which contains about 23.46% of similar normative feldspar, shows a few per cent present in the section. From a somewhat cursory study of the sections of the analysed specimens from this on to No. 12, it appears that the amount of visible (modal) feldspar is uniformly less than the calculated

amount. It would seem, therefore, that the limit of solubility of albite (with a little anorthite) in diopside-jadeite is somewhat less than 20% by weight of the mixture. Further study of this somewhat interesting point is being undertaken.

The normative mineral relations of the Middle American jades are shown in the accompanying figure 1, in which the apices are the points for 100% (by weight, not molecular) of albite (including anorthite), jadeite, and diopside; the loci are those of the specimens whose analyses are given in

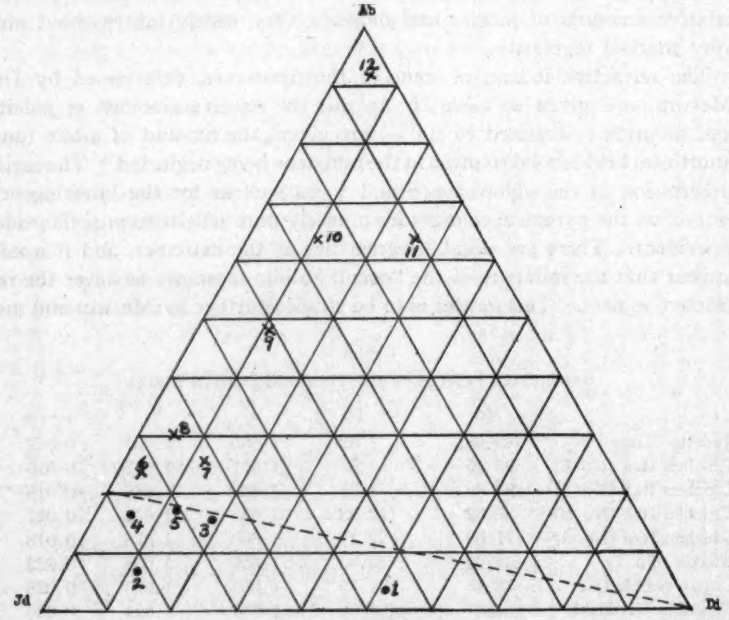

FIGURE 1
Mineral composition of jades

table I (except No. 13). There appears to be a marked tendency to cluster about the general composition: 70 jadeite, 10 diopside, 20 albite; and the greater number are strung along and rather near the albite-jadeite side. It is noteworthy that none of them fall on this border line, indicating the constant presence of diopside; while two fall almost on the line of equal amounts of jadeite and diopside, and only one (the Copan pebble) falls well within the dominantly diopside-albite part of the triangle.

In figure 1 the jades without visible feldspar are indicated by dots, while those with feldspar visible in the section are shown by crosses. The areas

of the two are separated by a dashed line, which extends from the diopside apex (on the basis of Bowen's experimental observation that typical diopside is incapable of holding feldspar in solution), to a point on the Ab-Jd border, at about 80 Jd–20 Ab. Some other analyses might have been plotted, which harmonize with those shown. The straightness and exact position of the dividing line are somewhat conjectural, but more jades which appear to fall near it, are being studied chemically and optically, so as to determine its position more accurately.

This series of rocks belongs evidently to the general group of jadeite jades, but differs from the usual jadeite of Burma and other sources of Chinese jade in two important particulars: the constant presence of large or considerable amounts of diopside with the jadeite in the pyroxene; and the presence of much albite in most of the series, either wholly in solid solution in the pyroxene, or partly so ("occult") and in part separately crystallized. The series, thus, extends from pure tuxtlite (diopside-jadeite, 1:1) to nearly pure albite. Further discussion of these points cannot be undertaken here, for lack of space; but it is suggested that this series shows such well-marked general and serial characters as to be deserving of a special name. That here proposed is *mayaite* which recalls the ancient nation who used and valued it, and which may distinguish it from the more widely known Burmese jadeite jades (with little or no diopside or albite). Purely jadeite jade seems to be unknown from, or at least is of very rare occurrence in, Middle America.

From these petrological considerations, the conclusion is inevitable that the material of the Central American and Mexican "jade" objects is of American and not Asian provenance. Although no American locality is yet known where *mayaite* occurs in situ, yet there are numerous specimens which are either clearly pebbles or show traces of pebble surfaces. But this is not the place for archaeological discussion. It may only be said, in conclusion, that various considerations regarding the occurrence of igneous rocks in different parts of the North American continent, as well as of the occurrence of Middle American jade objects, lead me to the belief that: (1) the jades come unquestionably from either Mexico or from Central America, or from both, and that (2) there are probably two centers of supply, both near the west (Pacific) coast, one probably in Oaxaca or Guerrero, and the other probably in Guatemala.

[1] The term "Middle America" is used to include both Mexico and Central America.
[2] H. S. Washington, *Proc. U. S. Nat. Mus.*, **60**, Art. 14, 1922.
[3] Clarke and Merrill, *Proc. U. S. Nat. Mus.*, **11**, 1888 (122, 124).
[4] H. S. Washington, *J. Wash. Acad. Sci.*, **12**, 1922 (387).
[5] N. L. Bowen, *Amer. J. Sci.*, **40**, 1915 (161).

[6] The birefringence, γ-α, of jadeite is uniformly given in the literature as 0.029, following apparently the early determination of Lévy and Lacroix (*Les Minéraux des Roches*, 1888 (266)). This value is certainly too high for pure jadeite, and the value 0.012, determined by Merwin on an analysed and crystallographically described crystal of pure jadeite, should replace it.

[7] Merwin, in Washington, *Proc. U. S. Nat. Mus.*, **60**, Art. 14, 1922 (9).

[8] Wright and Larsen, *Amer. J. Sci.*, **37**, 1909 (33).

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*ON A COMPARISON OF THE RELATIVE SENSITIVENESS OF TELEPHONES**

By Carl Barus

Department of Physics, Brown University

Communicated, September 20, 1922

In the earlier papers I reduced the fringe deflections s, of the interferometer U-gauge, produced by telephone currents of different effective strength by the simple relation sr = const., where for a fixed effective voltage, the resistance r in the telephone circuit is very large compared with the inductance, etc. The telephone mouthpiece is connected with one or both shanks of the U-gauge by a quill tube, simple or branched, respectively, each branch containing a pinhole probe suitably directed. Thus s measures the acoustic pressure evoked when the telephone sounds.

The expression given is applicable only within the narrow range of s values and low frequencies for which it was used. For wider ranges an exponential relation, which may be written

$$\log s = \log s_0 - r/r_0$$

where s_0 and r_0 are constant, fits the observations as accurately as the fringes can be read. To give examples chosen at random for a single telephone, branched connection, salient and reëntrant pinholes (s_0 = 399 fringes, r_0 = 6380 ohms)

$r \times 10^{-3}$	4	5	6	7	8	ohms
s observed	95	65	45	32	22	fringes
s computed	94	65	46	32	22	fringes

and the same for low ranges of s(s_0 = 190, r_0 = 8620)

$r \times 10^{-3}$	8	9	11	14	18	ohms
s observed	22	17	10	5	1	fringes
s computed	22	17	10	5	2	fringes

Similarly in case of binaural telephones, one corresponding to each shank of the U-tube, $s_0 = 152$, $r_0 = 49{,}500$, the results were, for example,

$r \times 10^{-3}$	14	23	33	43	ohms
s observed	105	65	41	25	fringes
s computed	103	66	41	25	fringes

If the effective voltage is constant and $s = 1$ in the equation, $r' = r_0 \log s_0$, where r' is the resistance needed to reduce the deflection to one fringe. Hence r' is an index of the sensitiveness of telephones, all operating in the given circuit successively. A number of tests so made ranged from $r' = 3 \times 10^3$ for common telephones to $r' = 10^5$ for high resistance telephones.

The meaning of the equation given is not so easily determined, since the acoustic pressure is produced through the intermediation of the telephone mechanism. If we suppose that, within limits, the amplitude a of the air waves produced at the plate is proportional to the amplitude of the current in the telephone, the exponential equation would then belong to the acoustic pressure p, and may be written, $p = p_0 e^{a_0/a}$. This seems to me probable, as it would explain the absence of registry in case of the pinhole resonator, for instance, when the sounding organ pipe is over 5 meters distant; i.e., the rapid decline of the fringe deflection with the distance of the source of sound may thus be quantitatively evaluated.

* Advance note from a report to the Carnegie Institution of Washington, D. C.

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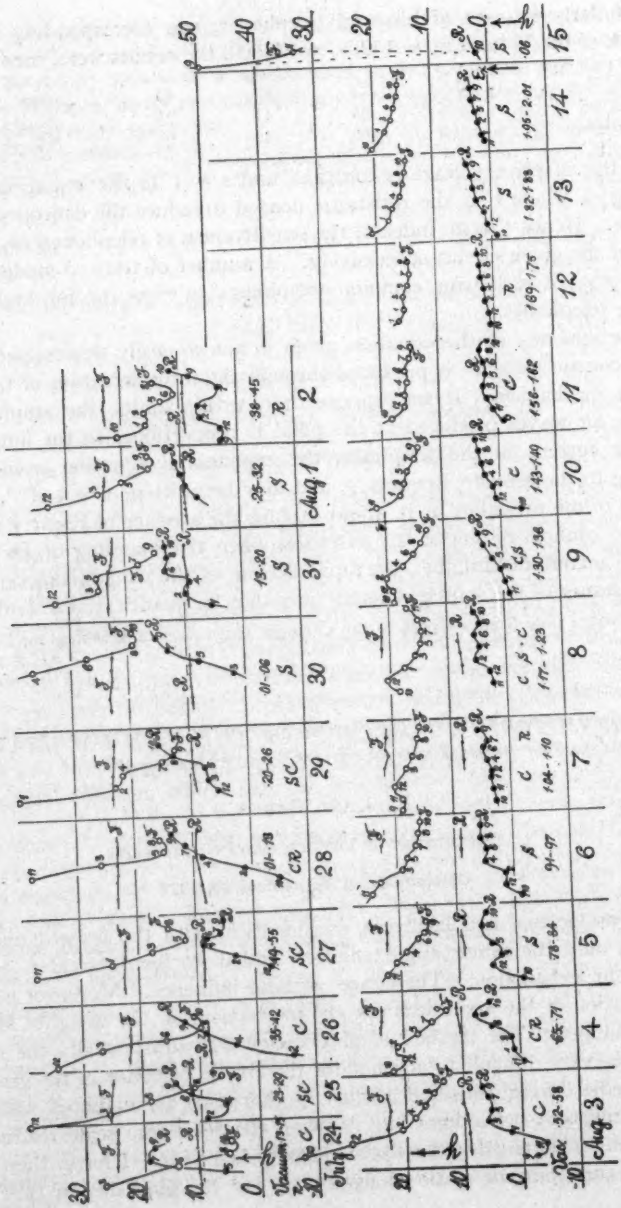
*THE EQUILIBRIUM POSITIONS OF THE VACUUM GRAVITATION NEEDLE IN 1921 AND 1922**

BY CARL BARUS

DEPARTMENT OF PHYSICS, BROWN UNIVERSITY

Communicated, September 20, 1922

The apparatus used last year was identical with the present one, excepting only the minor appurtenances, added to secure greater convenience in exhaustion. These are without influence. Moreover neither the location of the apparatus nor the mechanism of the weights M had been changed. The method of observation was substantially the same. It is thus very difficult to account for the erratic behavior of the gravitation needle during July and August of this year, as compared with the nearly constant deflections (Δy, y being the telescopic scale reading in centimeters) obtained under like conditions last year. I have, therefore, in the figure, put the static elongations y for the same day in 1921 and



1922 together, successive observations of the position of equilibrium (about one hour apart) being inserted equidistant horizontally. Data for 1922 are distinguished by little circles with the nearest hour number of the observation attached. Data for 1921 are given in points, in corresponding positions, for easy comparison. F denotes that the attracting weight M on the right end of the needle is in front of it. R, that M is to the rear of the needle. Complete sets of observations (i.e. equilibrium curves F and R) are given between about 10 A.M. and 10 P.M. for each year from July 24 to August 8 and to Aug. 15 in 1922. The exhaustion of the case in the morning and at night is shown on each day for 1922 only, as the data were not taken in 1921. In the same place C refers to cloudy, C' to partly cloudy weather, S to sunshine or a clear day and R to rain. It is seen at a glance that the variations of the position of equilibrium y in the lapse of time, are of a different order in 1922, from their approximate constancy in the given scale, in 1921. The contrast is startling with nothing easily apparent to account for it, unless it is traceable to a difference in the respective va ua. For the exhaustions, as a whole, were thought to have been of about the same order of value in both years. In 1922 there was fresh exhaustion on July 24, 28 and 30. Thereafter I left the apparatus with its slight leak (12 mm. per day), to itself, until August 15. In 1921 there were no intermediate exhaustions and unfortunately the McLeod gauge was not attached to the case.

The figure shows that all observations have a period of one day (24hr). Consequently the variations cannot be contributed by anything within the laboratory, though their enhancement is possible. They must in other words be originally meteorological, and due to solar radiation. Immediately after exhaustion (as on July 28, 30) there are apt to be large deflections, probably showing that the air within the case, though very rare, has nevertheless an appreciable cooling or other effect. For this reason I ceased to make exhaustions after July 30, and the observations, possibly for this contributory reason among others, gradually grew smoother and more normal. Any slight cooling of the inside of the case would tend to exaggerate the day time meteorological effect. If the cooling due to an exhaustion from 1 mm. to .001 mm. of pressure were instantaneous one would estimate that about .01 ca. would be abstracted from the air within the case. This is largely supplied of course by the massive case and the exhaustion is necessarily very gradual. Nevertheless the effect on the needle is always very marked. One cannot observe with the pump running, as the needle persistently clings to the wall of the case.

An inspection of the equilibrium curves in July, 1922, shows that F and R are quite liable to intersect somewhere between 6 and 8 P.M. which means that the radiant forces are in excess (Δy negative) and they may

thereafter considerably overbalance the gravitational pull. This happens when the temperature-time coefficient $d\theta/dt$ is negative. When the latter is positive, gravitation and the radiant forces coöperate. It is seen, moreover, that the higher exhaustions (a few tenths mm. and less), are (like the plenum) favorable to strong radiant forces; whereas in moderate exhaustions (a few millimeters) the radiant forces tend to a minimum, so that the night observations become more and more trustworthy. This may be inferred from the gradual simplification of the F, R curves between Aug. 1 and August 14. In corroboration, the apparatus was again exhausted on Aug. 14 and tested about 12 hours later on the morning of Aug. 15. It is seen that the strong radiant forces have reappeared.

* Advance note from a report to the Carnegie Institution of Washington, D. C.

FURTHER MEASUREMENTS OF STELLAR TEMPERATURES AND PLANETARY RADIATION

By W. W. Coblentz

Bureau of Standards, Washington, D. C.

Communicated September 21, 1922

1. *Introductory Statement.*—In a previous communication[1] estimates were given of the temperatures of 16 stars as determined from their spectral energy distribution which was determined by means of a new spectral radiometer, consisting of a series of transmission screens and a vacuum thermocouple.

By means of these screens, which, either singly or in combination, had a uniformly high transmission over a fairly narrow region of the spectrum and terminating abruptly to complete opacity in the rest of the spectrum, it was possible to obtain the radiation intensity in the complete stellar spectrum as transmitted by our atmosphere.

Not being equipped at that time for making radiometric measurements on the Sun, the effective temperature of which is known with some degree of accuracy, and hence could be used as a standard, an estimate of the effective temperature of a star was obtained by two methods, one of which consisted in using a solar type star (α Aurigae, Class G o) as a standard. This seemed permissible in view of the fact that the observed temperature (6000° K) of α Aurigae was found to be in close agreement with that assigned to the Sun.

The object of the present communication is to report a verification of the above-mentioned stellar radiation measurements, by similar measure-

ments recently made on the Sun; also to give some measurements of the radiation from the major planets.

2. *Apparatus and Procedure.*—The apparatus used in this work consisted of the 40-inch reflector of the Lowell Observatory, to which was attached a special radiometer mentioned in a previous issue of these PROCEEDINGS,[1] and fully described elsewhere.[2] The thermocouples were new and were made of fine bismuth wire, with receivers 0.2 mm. in diameter.

The crucial part of this investigation consisted in obtaining an exact measurement of the water cell transmission of the direct sunlight. For this purpose, the great reflecting mirror and the end of the telescope tube were covered. Parallel light, entering a small opening in the cover of the telescope tube, fell upon a small, plane, silvered mirror, placed in the axis of the tube, and from thence reflected upon the diagonal mirror to a thermopile which was especially constructed for these measurements.

3. *Experimental Data.*—The solar radiation data are given in table 1, in which column 1 gives the spectral range, and columns 2 and 4 give the spectral radiation components (A.M. and P.M.) of the Sun (also of α Aurigae, column 6) corrected for all losses except atmospheric absorption. The corresponding temperatures are given in columns 3, 5 and 7.

TABLE 1

SPECTRAL RADIATION COMPONENT OF THE SUN AND OF α AURIGAE IN PER CENT OF THE TOTAL

	SUN				α AURIGAE	
	A.M.		P.M.			
Spectral range. μ = 0.001 mm.	Spectral radiation component	Temperature °K	Spectral radiation component	Temperature °K	Spectral radiation component	Temperature °K
0.30μ to 0.43μ	25.0%	5670	25.6%	5750	18.4%	5000?
0.43μ to 0.60μ	22.1	6500	19.0	5760	18.2	5800
0.60μ to 1.40μ	31.8	5860	35.0	5050?	30.6	6000
1.40μ to 4.10μ	20.6	6060	20.1	6140	26.4	5200?
4.10μ to 10.00μ	0.5	....	0.8	....	6.4	....
Average		6000° K		5900° K		5900° K

These data are interesting in showing a close agreement between the spectral radiation components of the Sun and α Aurigae in the visible spectrum (0.43μ to 0.6μ) which is in agreement with the visual and photographic observations. However, in the ultra-violet the radiation component of α Aurigae is less, and in the infra-red it is greater (owing partly to the low-temperature radiation from the companion star), than that of the Sun. The lower value in the ultra-violet might be owing, in part, to the larger air mass traversed and hence to the greater amount of scattering of the incoming radiations. But this does not explain, and is inconsistent with, the higher values observed in the infra-red, in many of the stars previously measured, in spite of the greater air mass traversed.

No doubt this is owing partly to experimental errors; but some of it may be attributed to the low (selective) infra-red emission of the Sun, the earth's atmosphere being quite transparent to radiations greater than 7μ.

For the purposes of this investigation the agreement between the observed temperatures of α Aurigae and the Sun is considered satisfactory, verifying the previous measurements of stellar temperatures which range from 3000° K for red, class-M stars, to 12,000° K for blue, class-B stars.

Planetary Radiation.—The thermal radiation emitted from a planet as a result of warming by exposure to solar radiation, including heat which may be radiated by virtue of a possible high internal temperature of the planet itself, is essentially of long wave-lengths 7μ to 12μ. Hence, by means of a 1 cm. cell of water, interposed in the path of the total radiation emanating from the planet, this long wave-length radiation can be separated from the reflected solar radiation, and in this manner a measurement obtained of the energy reradiated. If there is planetary radiation, the water cell transmission will be less than that of the direct solar radiation.

The observed transmission of planetary radiation through a cell of water 1 cm. in thickness is given in table 2. This table shows that the

TABLE 2

TRANSMISSION OF PLANETARY RADIATION THROUGH A CELL OF WATER 1 CM. IN THICKNESS; ALSO THE PER CENT OF PLANETARY RADIATION EMITTED

Object	Date June, 1922 MST	No. of sets of measurements	PER CENT TRANSMISSION Deviation from mean	Mean value 1922	Mean value 1914	Planetary radiation in per cent of the total
Sun	17th; 11:30 A.M.	5	1.2%	69.3	....	..
	4:15 P.M.	4	0.5	69.8	....	..
Jupiter						
Central disk	14th; 10:15 P.M.	2	1.4	69.7	65.6	..
Southern hemisphere	10:20 P.M.	2	3.0	66.8	65.8	..
Venus	15th; 8:00 P.M.	2	2.3	66.3	59.0	5
Saturn	14th; 9:00 P.M.	3	1.1	60.0	55	15
Mars	15th; and 18th	.	6.0	50.3	....	30
Moon	.....	.	....	....	14.7	80

water cell transmission of the total radiation from Jupiter is practically the same as that of the direct solar radiation. From this it appears that the outer atmosphere of Jupiter does not radiate appreciable energy as the result of warming by solar rays and that the atmosphere is sufficiently thick and opaque to trap all the energy reradiated as the result of warming of its interior by solar radiation, or by internal heating.

The radiometric measurements on Venus, Jupiter and Saturn are in good agreement with similar measurements made at Mt. Hamilton, Calif.,

in 1914, showing a decidedly lower transmission of radiation through the water cell, in the case of Venus and Saturn.

The intensity of the planetary radiation increases with decrease in the density of the surrounding atmosphere and (as interpreted from the water cell transmissions) in per cent of the total radiation emitted, is as follows: Jupiter (O), Venus (5), Saturn (15), Mars (30) and the Moon (80).

The water cell transmission of the radiations from the southern (50.6%) and northern (53.1%) hemispheres of Mars should be and is higher than that of the radiations emanating from the equatorial (47.3%) region, owing to the depletion of the reradiated energy by the greater air mass. Moreover, the intensity of the planetary radiation from the northern hemisphere of Mars was found to be less than from the southern hemisphere. This is to be expected in view of the observed cloudiness over the northern hemisphere which is usually the brighter and is approaching the winter season, and hence is at a lower superficial temperature.

These data were obtained through the generosity of the Lowell Observatory, Flagstaff, Arizona, who financed this research. Dr. C. O. Lampland again kindly operated the telescope and it is intended to publish the complete results in a joint paper on the measurements of planetary radiation and their astronomical significance.

As already stated, the object of the present note is primarily to place on record a verification of the results, obtained a year ago, on stellar temperatures.

[1] *Proc. Nat. Acad. Sci.*, **8**, 1922 (49–53).

[2] *Bureau of Standards Scientific Paper*, No. **438**, 1922.

PROOF OF A THEOREM DUE TO HEAVISIDE[1]

BY H. A. LORENTZ

HAARLEM, HOLLAND

Communicated August 7, 1922

In the *Electrical Papers*, Vol. 2, p. 412, Heaviside states: "The whole work done by impressed forces suddenly started exceeds the amount representing the waste by Joule-heating at the final rate (when there is any), supposed to start at once, by twice the excess of the electric over the magnetic energy of the steady field set up."

Consider a system of bodies, either conductors or dielectrics, or, for the sake of generality, both at the same time, and denote by **E** the electric force, by **H** the magnetic force, by **D** the dielectric displacement, by **C**

the current of conduction, by **B** the magnetic induction. We shall suppose the matter to be isotropic, so that at each point we have to introduce only one ϵ (dielectric constant), one permeability μ, and one conductivity σ.[2] These quantities may change from place to place; they, as well as **E, H,** etc., are considered as continuous functions. If there are different bodies, transition layers in which the properties gradually change from one body to the other, are supposed to exist at the common boundary. (This is only for the purpose of mathematical convenience; the thickness of the transition layer can be supposed to diminish indefinitely, and so we can pass to the limiting case of a sharp demarcation.) Let there be two kinds of impressed forces ("electromotive" forces), the one $\mathbf{F}_1$ producing dielectric displacement, the other $\mathbf{F}_2$ producing conduction currents.[3] The meaning of this is that the dielectric displacement is determined by

$$\mathbf{D} = \epsilon(\mathbf{E} + \mathbf{F}_1) \tag{1}$$

and the conduction current by

$$\mathbf{C} = \sigma(\mathbf{E} + \mathbf{F}_2) \tag{2}$$

The forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are again continuous functions of the coördinates. Each of them may be confined to a limited space and these two spaces may lie one outside the other or they may overlap more or less. At a definite point there is but one **E**, the same in (1) and (2).

In addition to (1) and (2) we have

$$\mathbf{B} = \mu\mathbf{H} \tag{3}$$

$$\text{curl}\,\mathbf{H} = \frac{1}{c}(\dot{\mathbf{D}} + \mathbf{C}) \tag{4}$$

$$\text{curl}\,\mathbf{E} = -\frac{1}{c}\dot{\mathbf{B}} \tag{5}$$

$$\text{div}\,(\dot{\mathbf{D}} + \mathbf{C}) = 0 \tag{6}$$

$$\text{div}\,\mathbf{B} = 0 \tag{7}$$

Electric energy per unit of volume $\mathbf{D}^2/2\epsilon$.

Magnetic energy per unit of volume[4] $\frac{1}{2}(\mathbf{B}.\mathbf{H})$.

Joule-heat per unit of volume and unit of time $\mathbf{C}^2/\sigma$.

Work of $\mathbf{F}_1$ per unit of volume and unit of time $(\mathbf{F}_1.\dot{\mathbf{D}})$.

Work of $\mathbf{F}_2$ per unit of volume and unit of time $(\mathbf{F}_2\,\mathbf{C})$.

We shall suppose that the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are started during an infinitely short interval of time Δt, ending at the instant $t = 0$; from this latter onward they remain constant. Finally there will be a steady state. In what follows the integrations with respect to the time are extended up to

an instant t at which the final state has been practically reached. The final values of $\mathbf{D}$, $\mathbf{C}$, etc. are denoted by $\overline{\mathbf{D}}$, $\overline{\mathbf{C}}$, etc.

The main point in the following reasoning is that, when the interval Δt is infinitely short, the values of $\mathbf{D}$ and $\mathbf{B}$ at the instant $t = 0$ are likewise infinitely small, so that we may reckon with $\mathbf{D} = 0$, $\mathbf{B} = 0$ for $t = 0$; moreover, at $t = 0$ the current $\mathbf{C}$ cannot be infinite and we may neglect the work of the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ during the interval Δt. In the time-integral representing the work (up to the time t), we may take $t = 0$ for the lower limit.

Proof.—During the interval Δt the finite forces $\mathbf{F}_1$ and $\mathbf{F}_2$ cannot produce an infinite $\mathbf{E}$. Therefore, according to (5) $\dot{\mathbf{B}}$ can at the best have a finite value. But then (because at the beginning of Δt, $\mathbf{B} = 0$) $\mathbf{B}$ must be infinitely small until $t = 0$. The same is true of $\mathbf{H}$ (because connected with $\mathbf{B}$ by (3)). Therefore, on account of (4), $\dot{\mathbf{D}} + \mathbf{C}$ must be infinitely small. But, as (2) shows, $\mathbf{C}$ cannot be infinite; therefore $\dot{\mathbf{D}}$ also cannot be infinite, and $\mathbf{D}$ must be infinitely small at $t = 0$.[5] We can conclude now that both the integrals $\int(\mathbf{F}_1.\dot{\mathbf{D}})dt$ and $\int(\mathbf{F}_2.\mathbf{C})dt$ which represent the work of the electromotive forces, when taken over the infinitely small time Δt, have infinitely small values; indeed, neither $(\mathbf{F}_1.\dot{\mathbf{D}})$ nor $(\mathbf{F}_2.\mathbf{C})$ becomes infinite in the interval.

dS is the element of volume. Integrations with respect to dS are extended over the whole field, when necessary to infinite distance.

The electric energy in the final state is $U = {}^1/_2 \int \frac{\overline{\mathbf{D}}^2}{\epsilon} dS$, or, on account of (1)[6]

$$U = \frac{1}{2}\int(\{\overline{\mathbf{E}} + \mathbf{F}_1\}.\overline{\mathbf{D}})dS \qquad (8)$$

The magnetic energy in the final state is $T = {}^1/_2 (\overline{\mathbf{H}}.\overline{\mathbf{B}})dS$.

The Joule heat per unit of time in the final state is

$$\int \frac{\overline{\mathbf{C}}^2}{\sigma} dS,$$

for which we may also write[7] $\int(\mathbf{F}_2.\overline{\mathbf{C}})dS$.

If the Joule generation of heat started suddenly at $t = 0$ at the rate given by the last expression, the amount of heat developed until the time t would be[8]

$$W = \int_0^t dt \int (\mathbf{F}_2.\overline{\mathbf{C}})dS \qquad (9)$$

The work of the impressed forces up to the time t is given by

$$A = \int_0^t dt \int (\mathbf{F}_1.\dot{\mathbf{D}})dS + \int_0^t dt \int (\mathbf{F}_2.\mathbf{C})dS, \qquad (10)$$

where we may write for the first term $\int(\mathbf{F}_1.\bar{\mathbf{D}})dS$ because $\mathbf{F}_1$ is constant and $\int_0^t \dot{\mathbf{D}}dt = \bar{\mathbf{D}}$($\mathbf{D}$ being 0 for $t = 0$).

From (10) and (9):

$$A - W = \int(\mathbf{F}_1.\bar{\mathbf{D}})dS + \int_0^t dt \int(\mathbf{F}_2.\{\mathbf{C} - \bar{\mathbf{C}}\})dS \qquad (11)$$

In virtue of (8):

$$\int(\mathbf{F}_1.\bar{\mathbf{D}})dS = 2U - \int(\bar{\mathbf{E}}.\bar{\mathbf{D}})dS,$$

and by (2), applied to the final state, $\mathbf{F}_2 = \bar{\mathbf{C}}/\sigma - \bar{\mathbf{E}}$. Substituting these values in (11) we find

$$A - W = 2U + \int_0^t dt \int\left(\frac{1}{\sigma}\bar{\mathbf{C}}.\{\mathbf{C} - \bar{\mathbf{C}}\}\right)dS - \\ - \int(\bar{\mathbf{E}}.\bar{\mathbf{D}})dS - \int_0^t dt \int(\bar{\mathbf{E}}.\{\mathbf{C} - \bar{\mathbf{C}}\})dS \qquad (12)$$

Now, we may write

$$\bar{\mathbf{D}} = \int_0^t \dot{\mathbf{D}}dt,$$

and we can therefore combine the two last terms of (12) into

$$-\int_0^t dt \int(\bar{\mathbf{E}}.\{\dot{\mathbf{D}} + \mathbf{C} - \bar{\mathbf{C}}\})dS \qquad (13)$$

We have by (5) and (6)

$$\text{curl } \bar{\mathbf{E}} = 0, \qquad \text{div } (\dot{\mathbf{D}} + \mathbf{C}) = 0, \qquad \text{div } \bar{\mathbf{C}} = 0;$$

hence (13) vanishes[9] and (12) reduces to

$$A - W = 2U + \int_0^t dt \int\left(\frac{1}{\sigma}\bar{\mathbf{C}}.\{\mathbf{C} - \bar{\mathbf{C}}\}\right)dS \qquad (14)$$

Transformation of the last term: Substitute $\bar{\mathbf{C}} = c \text{ curl } \bar{\mathbf{H}}$ (form. (4) applied to final state), $\mathbf{C} - \bar{\mathbf{C}} = \sigma(\mathbf{E} - \bar{\mathbf{E}})$ (form. (2) combined with corresponding equation for final state), integrate by parts, and finally replace curl $\mathbf{E}$ by $-\dot{\mathbf{B}}/c$ and curl $\bar{\mathbf{E}}$ by 0.

$$\int_0^t dt \int\left(\frac{1}{\sigma}\bar{\mathbf{C}}.\{\mathbf{C} - \bar{\mathbf{C}}\}\right)dS = \int_0^t dt \int (c \text{ curl } \bar{\mathbf{H}}.\{\mathbf{E} - \bar{\mathbf{E}}\})\, dS^{10}$$

$$= \int_0^t dt \int(c\bar{\mathbf{H}}.\text{curl } \{\mathbf{E} - \bar{\mathbf{E}}\})dS = -\int_0^t dt \int(\bar{\mathbf{H}}.\dot{\mathbf{B}})dS$$

$$= -\int dS \int_0^t (\bar{\mathbf{H}}.\dot{\mathbf{B}})dt \qquad (15)$$

Now, since $\overline{\mathbf{H}}$ does not depend on t, and $\mathbf{B} = 0$ for $t = 0$,

$$\int_0^t (\overline{\mathbf{H}}.\dot{\mathbf{B}})dt = (\overline{\mathbf{H}}.\overline{\mathbf{B}});$$

(15) therefore becomes

$$-\int(\overline{\mathbf{H}}.\overline{\mathbf{B}})dS = -2T$$

and (14)

$$A - W = 2(U - T) \qquad \text{q. e. d.} \qquad (16)$$

Remarks.—1. Both A and W depend on the choice of the time t the end of the (long) interval of time which we considered. But the *difference* $A - W$ is independent of that choice. Indeed, in the steady state, the work of the impressed force per unit of time is equal to the generation of heat, equally per unit of time, so that, when the interval of time is lengthened, A and W increase by equal amounts.

When, in the final state, there are no currents (but only electric charges) $W = 0$. In this case A is independent of the choice of t.

2. W is not the quantity of heat that is *really* generated. This latter quantity W' is given by

$$W' = \int_0^t dt \int \frac{\mathbf{C}^2}{\sigma} dS.$$

3. In the interval of time during which the steady state is reached there will in general be a *radiation* of energy from the system outward. Let the total amount of the radiated energy (to be calculated by applying Poynting's rule to the flow through an infinitely great sphere) be R. Then, by the law of conservation of energy

$$A = W' + U + T + R.$$

Comparing this with (16) one finds

$$W - W' = 3T - U + R.$$

4. The theorem expressed by (16) can easily be verified for simple cases, when all quantities involved can be completely calculated.

(*a*). A linear circuit with resistance r and self-inductance L, no capacity. Let the electromotive force $\mathbf{E}$ be suddenly applied at $t = 0$, remaining constant ever afterwards

The current is given by $i = E/r.(1 - e^{-rt/L})$, and one finds

$$A = \int_0^t Eidt = \frac{E^2}{r}t - \frac{E^2L}{r^2} \quad (t \text{ very great}),$$

$$W = \frac{E^2}{r}t, \quad T = \frac{LE^2}{2r^2}, \quad U = 0.$$

(*b*). A dielectric with the constant ϵ fills all space. At time $t = 0$ an electromotive force $\mathbf{F}_1$ is suddenly applied. It has the same direction and magnitude at all points inside a sphere and is limited to that part of space. The steady state is now easily calculated; likewise the amount of energy R, that is radiated.

5. The theorem mentioned in G. W. Pierce, *Electrical Oscillations*, p. 40, is closely connected with Heaviside's general theorem.

Consider a circuit containing a condenser and in series with it a self-induction and resistance. In this case $T = 0$, $W = 0$, so that (16) becomes $A = 2U$. But, by the law of energy ($R = 0$), $A = W' + U$. Therefore $W' = U$, which is Pierce's statement.

[1] This general theorem stated without proof by Heaviside, was useful to Mr. Charles Manneback in his thesis for the doctorate in science at the Massachusetts Institute of Technology, June 1922: *Radiation from Transmission Lines* (as yet unpublished). As Dr. Manneback could not locate, in the textbook or periodical literature, a satisfactory proof of the theorem in all its generality, he appealed for help to Professor Lorentz who kindly sent the demonstration. It has seemed as though the treatment of the theorem by Dr. Lorentz with his comments upon it might be of wide interest, particularly on account of possible practical applications, and he has assented to its publication.—EDITOR.

[2] One can just as well (somewhat greater complication of the formulae) consider crystalline substances.

[3] One could add a "magneto-motive" force, but for the sake of simplicity we shall not do so.

[4] $(\mathbf{A}.\mathbf{B})$ is the scalar product of the vectors $\mathbf{A}$ and $\mathbf{B}$; $(\mathbf{A}.[\mathbf{B} + \mathbf{C}])$ the scalar product of the vectors $\mathbf{A}$ and $\mathbf{B} + \mathbf{C}$. Of course when two vectors have the same direction (as $\mathbf{B}$ and $\mathbf{H}$) we can as well take the product of their magnitudes.

[5] What is said here of $\mathbf{D}$ does not apply to the electric force. When, during the interval Δt, $\mathbf{D}$ is infinitely small (practically 0), *eq.* (1) shows that $\mathbf{E}$ is nearly $-\mathbf{F}_1$, a finite value. Note that we have drawn our conclusions from the fact that the equations contain $\dot{\mathbf{B}}$ and $\dot{\mathbf{D}}$. They do not contain $\dot{\mathbf{E}}$.

[6] We need not write $\overline{\mathbf{F}}_1$ and $\overline{\mathbf{F}}_2$, because $\mathbf{F}_1$ and $\mathbf{F}_2$ are constant from $t = 0$ onward.

[7] This formula expresses the fact that in the final state, when both U and T remain constant, the Joule heat is equal to the work of $\mathbf{F}_2$. (The work of $\mathbf{F}_1$ is 0, because in the final state $\dot{\mathbf{D}} = 0$.) One can deduce one form from the other by using $\overline{\mathbf{C}} = \sigma(\overline{\mathbf{E}} + \mathbf{F}_2)$. $\int (\overline{\mathbf{E}}.\overline{\mathbf{C}})dS = 0$, because div $\overline{\mathbf{C}} = 0$ and curl $\overline{\mathbf{E}} = 0$.

[8] Since $(\mathbf{F}_2.\overline{\mathbf{C}})$ is independent of t we could just as well write $t.\int (\mathbf{F}_2.\overline{\mathbf{C}})dS$, but in what follows, the form (9) is to be preferred.

[9] First extend the integration to the space within a sphere Σ around O, the radius r of which ultimately increases indefinitely. Since $\overline{\mathbf{E}}$ depends on a potential φ, we find by partial integration (putting $\dot{\mathbf{D}} + \mathbf{C} - \overline{\mathbf{C}} = \mathbf{Q}$) $\int (\overline{\mathbf{E}}.\mathbf{Q})dS = -\int (\text{grad } \varphi.\mathbf{Q})dS = -\int \varphi \mathbf{Q}_n d\Sigma + \int \varphi \text{ div } \mathbf{Q} dS$. ($\mathbf{Q}_n$ normal component.) The surface integral vanishes for $r = \infty$ because (when the total charge of the system is 0) φ decreases at least as $1/r^2$, whereas $\mathbf{Q}_n$ becomes 0.

[10] There the integration by parts leads to the surface integral

$c\int \{[\overline{\mathbf{H}}_y(\mathbf{E}_z - \overline{\mathbf{E}}_z) - \overline{\mathbf{H}}_z(\mathbf{E}_y - \overline{\mathbf{E}}_y)]\cos\alpha + \ldots.\}d\Sigma$ ($\cos\alpha$ ....direction constants of normal) to Σ and this vanishes for similar reasons as the surface integral of the preceding note.

THE STABILITY OF THE NORMAL AGE DISTRIBUTION[1]

BY ALFRED J. LOTKA

SCHOOL OF HYGIENE AND PUBLIC HEALTH, JOHNS HOPKINS UNIVERSITY

Communicated, September 13, 1922

There is a unique age distribution which, in certain circumstances,[2] has the property of perpetuating itself when once set up in a population. This fact is easily established,[3] as is also the analytical form of this unique *fixed* or *normal* age distribution.

More difficult is the demonstration that this age distribution is *stable*, that a population will spontaneously revert to it after displacement therefrom, or will converge to it from an arbitrary initial age distribution. Such a demonstration has hitherto been offered only for the case of small displacements,[4] by a method making use of integral equations. The purpose of the present communication is to offer a proof of stability which employs only elementary analytical operations, and which is readily extended to cover also the case of large displacements. This method presents the further advantage that it is molded in more immediate and clearly recognizable relation to the physical causes that operate to bring about the normal age distribution.

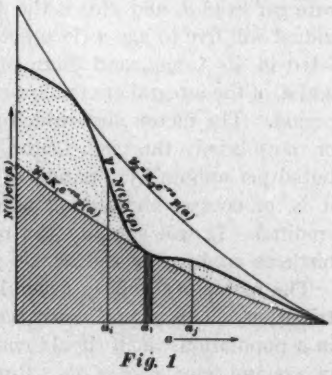


Fig. 1

Consider a population which, at time t has a given age distribution such as that represented by the heavily drawn curve in fig. 1, in which the abscissae represent ages a (in years, say), while the ordinates y are such that the area comprised between two ordinates erected at a_1 and a_2, respectively, represents the number of individuals between the ages a_1 and a_2.

If we denote by $N(t)$ the total population at time t, and if the ordinates of our curve are

$$y = N(t)\, c(t,a) \tag{1}$$

we have, evidently,

$$\int_{a_1}^{a_2} y\, da = N(t) \int_{a_1}^{a_2} c(t,a)da = N(t,a_1,a_2) \tag{2}$$

where $N(t,a_1,a_2)$ denotes the number of individuals living at time t and comprised within the age limits a_1 and a_2. We may speak of $c(t,a)$ as the coefficient of age distribution. It is, of course, in general a function of t, only in the special case of the fixed or self-perpetuating age distribution is $c(a)$ independent of the time.

Now, without assuming anything regarding the stability of the self-perpetuating age distribution, it is easy to show[2] that its form must be

$$c(a) = \frac{e^{-ra}p(a)}{\int_0^\infty e^{-ra}p(a)da} = be^{-ra}p(a) \tag{3}$$

where r is the real root of the equation

$$1 = \int_{a_i}^{a_j} e^{-ra}p(a)\beta(a)da \tag{4}$$

In this equation r is the natural rate of increase of the population, i.e., the difference $r = b - d$ between the birthrate per head b and the deathrate per head d, and $p(a)$ is the probability, at birth, that a random individual will live to age a (in other words, it is the principal function tabulated in life tables, and there commonly denoted by l_x). The limits a_i and a_j of the integral are the lower and upper age limits of the reproductive period. The factor $\beta(a)$, which might be termed the procreation factor, or more briefly the birth factor, is the average number of births contributed per annum by a parent of age a. (In a population of mixed sexes it is, of course, immaterial, numerically, to what parent each birth is credited. It will simplify the reasoning, however, if we think of each birth as credited to the female parent only.)

The factor $\beta(a)$ will in general itself depend on the prevailing age distribution. This is most easily seen in the case of extremes, as for example in a population which should consist exclusively of males under one year of age and females over 45. But, except in such extreme cases, $\beta(a)$ will not vary greatly with changes in the age constitution of the population, and we shall first develop our argument on the supposition that $\beta(a)$ is independent of the age distribution. We shall then extend our reasoning to the more general case of $\beta(a)$ variable with $c(t,a)$.

Referring now again to fig. 1, let two auxiliary curves be drawn, a *minor tangent curve* and a *major tangent curve*

$$y_1 = K_1e^{-ra}p(a) \quad (5) \qquad y_2 = K_2e^{-ra}p(a) \quad (6)$$

the constants K_1, K_2 being so chosen that the minor tangent curve lies wholly beneath the given arbitrary curve, except where it is tangent thereto, while the major tangent curve lies wholly above the given curve, except where it is tangent thereto.

The given arbitrary curve representing the age constitution of the population at time t then lies wholly within the strip or area enclosed between the minor and the major tangent curves.

Now consider the state of affairs at some subsequent instant t'. Had the population at time t consisted solely of the individuals represented by the lightly shaded area in fig. 1, i.e. the area under the minor tangent curve, then at time t' the population would be represented by the lower curve of fig. 2, whose equation is

$$y'_1 = K'_1 e^{-ra} p(a) \qquad (7)$$

$$= K_1 e^{r(t'-t)} e^{-ra} p(a) \qquad (8)$$

For the age distribution (5) is of the fixed form (3), and therefore persists in (7); on the other hand, given such fixed age distribution, the population as a whole increases in geometric progression,[3] so that $K'_1 = K_1 e^{r(t'-t)}$. In point of fact, we have left out of reckoning that portion of the population which in fig. 1 is represented by the dotted area. Hence, in addition to the population under the lower curve of fig. 2, there will, at time t', be living a body of population which for our present purposes it is not necessary to determine numerically. We need only know that it is some positive number, so that the curve representing the actual population at time t' must lie wholly above or in contact with the curve (8).

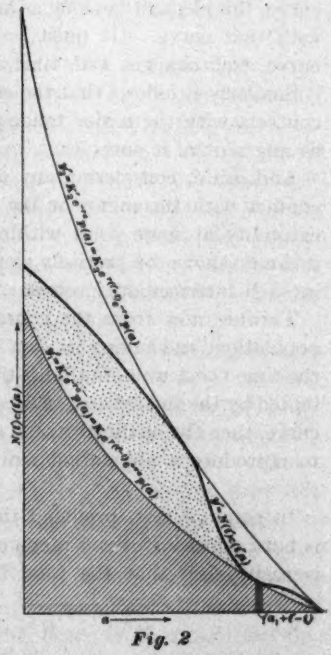


Fig. 2

The curves shown are intended to be interpreted only in a qualitative sense; so, for example, the increase in the ordinates in passing from fig. 1 to fig. 2 is very much exaggerated, to render it obvious to the eye.

By precisely similar reasoning it is readily shown that at time t' the actual curve lies wholly beneath or in contact with the curve

$$y'_2 = K_2 e^{r(t'-t)} e^{-ra} p(a) \qquad (9)$$

Hence at time t' the actual curve lies wholly within the strip comprised between the two curves (8) and (9).

Consider now an elementary strip, of width da, of the original population (shown heavily shaded in fig. 1), which at time t is in contact with the minor tangent curve. Let this con-

tact occur at age a_1, so that $y_t(a_1) = K_1 e^{-ra_1} p(a_1)$. At time t' the survivors of the individuals comprised in this elementary strip will be of age $(a_1 + t' - t)$, so that they will then be represented by a strip of width da and of altitude

$$y_{t'}(a_1 + t' - t) = K'_1 e^{-ra_1} p(a_1 + t' - t)$$
$$= K_1 e^{r(t'-t)} e^{-r(a_1+t'-t)} p(a_1 + t' - t)$$

From this it is seen that the elementary strip of population which at time t' contacts with the minor tangent curve (8) is built up of the survivors of the strip which at time t contacted with the minor tangent curve (5). In other words, if we follow up, by identity of individuals, the element of the population which at any instant contacts with the minor tangent curve, this element (so long as any part of it survives) continues in contact with that curve. (It must be remembered, however, that the tangent curve itself changes with time according to (5), (8).)

Similarly it follows that the element of population which at any instant contacts with the major tangent curve continues in this contact so long as any part of it survives.

And again, considering any element of the population which does not contact with the minor or the major tangent curves, but has its upper extremity at some point within the area enclosed between these curves, it can be shown by precisely similar reasoning that such element continues in such intermediate position.

Turning now from the consideration of the survivors of the original population, and taking in view the new population added by births since the time $t = t$, we note that if the original population had been that represented by the shaded area in fig. 1, i.e., by the area under the minor tangent curve, then the birthrate would at all times have been such as continuously to reproduce a population represented by the minor tangent curve (5), (8).

In point of fact, provided that contact with the minor tangent curve is not *continuous* over a range of ages equal to or greater than $a_j - a_i$, it is easily seen that the total birthrate

$$B = N \int_{a_i}^{a_j} c(a)\beta(a)da$$

is always greater (equality is here excluded) than that which would result from and in turn reproduce the age distribution represented by the minor tangent curve.

Similarly the total birthrate is always less than that which would result from a population and age distribution represented by the major tangent curve.

From this it is clear that, after the original population has once died out, the representative curve can never again contact with the original minor and major tangent curves, but must henceforth be separated from them everywhere by a finite margin (except, of course, where the several curves terminate upon the axis of a).

We may then begin afresh by drawing a new pair of tangent curves, lying within the original pair, and so on indefinitely, until the minor and major tangent curves coincide, and with them also coincides the actual curve of age distribution, which is then of the form

$$N(t)\, c(a) = Ke^{rt}e^{-ra}p(a)$$

which we recognize as the fixed age distribution (3), with

$$N(t) = Ke^{rt}\int_0^\infty e^{-ra}p(a)da$$

In this argument we have expressly excluded the case that the original age distribution curve contacts *continuously* with one of its tangent curves over a range greater than the reproductive period $(a_j - a_i) = A$. If such continuous contact occurs in the Oth generation over a range nA, where $a_j/A > n > 1$, then a simple reflection shows that in the next generation (the first) contact will occur over a range $(n - 1)A$, in the second generation over $(n - 2)A$, etc. A time is therefore reached (in practice very soon), when contact is over a period less than the period of reproduction A. After that the argument set forth above applies.

If the curve representing the original age distribution contacts with one of the tangent curves all the way from $a = 0$ to $a = a_j$, so that $a_j/A \leqq n$, then, of course, the fixed age distribution is practically established *ab initio*, or at any rate from the moment the original population above reproducing age a_j has died out.

It remains to consider the effect of variability in the form of $\beta(a)$ with changes in $c(t,a)$. Some such variability undoubtedly exists owing to the influence of the ages of the male and female constituents of the population upon the frequency of matings. We may, nevertheless, in this case also, define a minor and a major tangent curve (5), (6) in terms of the value of r given by equation (4); in order, however, to make this value determinate, it is now necessary to make some definite disposition regarding the form of $\beta(a)$, which is now variable. Merely for purposes of defining r, we shall suppose that the function $\beta(a)$ under the integral sign has that particular form which corresponds to the fixed age distribution.[5]

We cannot, however, now reason, as before, that the portion of the population represented by the shaded area in fig. 1 will, by itself, reproduce

its own form of age distribution. For its constitution as to sex will in general differ from that of the "normal" population with self-perpetuating age distribution. We must therefore consider three different possibilities:

1. The shaded area alone will produce a population exceeding at all ages the normal or fixed type continuation (8) of the original shaded area. Should this be the case, then the argument presented with regard to the case of invariable $\beta(a)$ holds *a fortiori*, so far as the minor tangent curve is concerned.

2. The shaded area alone will produce a population deficient at some or all ages, as compared with the normal type continuation (8) of the original shaded area. In that case two alternatives present themselves:

a. The deficiency is more than counterbalanced by the additional population produced by that portion of the original population which is represented by the dotted area in fig. 1. In this case also the original argument, so far as the minor tangent curve is concerned, applies essentially as before.

b. After the contributions from all parts of the population have been taken into account, there remains an unbalanced deficiency short of the population defined by (8), in the population resulting from that originally present. In such case the argument presented on the assumption of invariable $\beta(a)$ fails, and the population may move away from, not towards the age distribution(3). Stability of the fixed age distribution may not extend to such displacements as this.

Similar reflections apply, *mutatis mutandis*, as regards the major tangent curve.

If conditions (1) or (2) prevail with respect to the minor, and corresponding conditions with respect to the major tangent curve, then we can argue as in the case of invariable $\beta(a)$, that after expiration of the existing generation a new pair of tangent curves can be drawn, which will lie within the pair defined by (8), (9). And if conditions (1) or (2) still persist with reference to the new tangent curves, the same process of closing in these curves at the expiration of the current generation can be repeated, and so on, as in the argument first presented. Now conditions (1) or (2) will thus continue to prevail for each new pair of tangents, if, as the minor and major tangents close up, $\beta(a)$ approaches a limiting form. In such case, therefore, the age distribution defined by (3), (4) is stable even for displacements of any magnitude, provided always that conditions (1) or (2) prevail, as indicated.

[1] Papers from the Department of Biometry and Vital Statistics, School of Hygiene and Public Health, Johns Hopkins University, No. 71.

[2] These circumstances are: (*a*) an invariable life curve (life table); (*b*) an invariable ratio of male to female births; (*c*) an invariable rate per head of procreation at each year of age, for any given sex-and-age distribution in the population.

[3] Lotka, *Amer. J. Sci.*, **26**, 1907 (21); *Ibid.*, **24** (199); *J. Wash. Acad. Sci.*, **3**, 1913 (241, 289).

[4] Sharpe, F. R., and Lotka, A. J., *Phil. Mag.*, April 1911, 435.

[5] A certain ambiguity is introduced here by the fact that, with $\beta(a)$ of variable form, equation (4) might have more than one real root for r. In practice, however, in a human population at any rate, probably only one such root exists, or has any effect upon the course of events.